A MARKOVIAN APPROXIMATED SOLUTION TO A PORTFOLIO MANAGEMENT PROBLEM

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EXECUTIVE SUMMARY

This paper is about a computational optimisation method suitable for, and tested on, a dynamic portfolio problem.

If an economic agent maximises a utility function and considers what percentage of their wealth should be invested in an asset of an uncertain return, he (or she) solves a portfolio management problem. If the utility function is a sum of discounted future payoffs and the rate of investment can vary in time, then this is a dynamic portfolio optimisation problem.

Dynamic optimisation problems with uncertainties are usually modelled as continuous time and space stochastic control problems. They admit an analytical solution for specific formulations only. An alternative dynamic optimisation model is in the form of a Markov decision chain, discrete in time and space. Finite Markov decision chains are solvable through the dynamic programming technique albeit the solution time can rise exponentially in the number of the chain states. A suitable method for a discretisation of a stochastic control problem and conversion to a finite Markov decision chain is described in this paper. The method is implemented through a custom written suite of Matlab routines.

Approximately optimal decision rules for a “classical” (Merton) portfolio problem are computed. Extensions to this problem are also solved and optimal policies for a variable volatility portfolio and pension fund are calculated.

KEYWORDS: Computational economics; portfolio management; approximating Markov decision chains; weak Euler scheme.

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ABSTRACT. A portfolio management problem is approximated by a Markov decision chain. A weak Euler scheme is applied to discretise the time evolution of a portfolio and an inverse distance method is used to describe the transition probabilities. The approximating Markov decision problem is solved by value iteration. Numerical solutions of varying degrees of accuracy to some specific financial portfolio problems are obtained. A sample of a fund manager’s objective functions is analysed to tell which of them generates an acceptable Value-at-Risk.

1. Introduction

The purpose of this paper is to describe a numerical method capable of solving a class of stochastic optimal control problems, which includes portfolio management. The paper draws the idea of solving a continuous finite-horizon stochastic optimal control problem as a Markov decision chain from [6] and [5]. In this paper, a weakly consistent Euler discretisation scheme, used for the approximation of the stochastic process, and a scaling in the policy space have greatly improved the solutions, relative to those reported previously.

Optimal portfolio management can be modelled as a stochastic optimal control problem. In principle, one can usually solve a problem of this class by solving the Hamilton-Jacobi-Bellman (HJB) equation. However, this equation is often analytically insoluble and a numerical method has to be applied. This involves a discretisation scheme. (For computational methods used in financial engineering see, for example [10] and, for a review, [16].)

The Kushner [8] approach is an efficient discretisation scheme, in which the state-space and time steps remain related. Implementations of this approach to infinite horizon decision problems have been successful [11], even in the case of stochastic differential games [2].

In [7], [6] and [5] a simple approach was introduced that produced numerical solutions to a few finite-horizon stochastic optimal control problems. Instead of looking for a solution to the HJB equation, as in the Kushner approach, a Markov decision chain, discrete in time and space, was solved. This is a more elementary exercise: instead of looking for a numerical solution to a second-order partial differential equation (HJB), a first order difference equation (Bellman’s) needs to be solved.

In this paper, the original continuous optimal control problem is discretised to produce a Markov decision chain. A method of approximating the continuous noise by a discrete valued noise is applied. Value iteration is used to solve the Bellman equation for the Markov decision chain thus obtained. This “simple” Markovian approximation method was directly applied in [7] and [6] to estimate the discounted profit of stochastic resource utilisation. Encouraging results were
reported; in particular, a good level of agreement of numerical solutions with the existing solutions (see [10]) was achieved and a sensible degree of computational complexity of the method was observed. In [5], the same method was used for the solution to the classical portfolio management problem (see [1]). While the utility measures of the approximating and original problems were similar, there were some discrepancies in the policy profiles, in these earlier papers. These are overcome in this paper through a scaling in the policy space and a weakly consistent discretisation scheme of the Ito diffusion process.

The emphasis of the paper is on the solution method. However, a few financial engineering problems, difficult to solve analytically, will be solved numerically in this paper. In particular, rules will be computed for non stationary\(^1\) and constrained\(^2\) portfolio problems. A sample of a fund manager's objective functions is analysed to tell which of them generates an acceptable Value-at-Risk.

The rest of this paper is organised as follows. In Section 2 the Markovian approximation method from [7] is modified through use of weakly consistent (Euler) 2-value noise discretisation, rather than an intuitively motivated 3-value noise discretisation scheme used in [7], [6] and [5]. The method is applied, in Sections 4-5, to a classical optimal portfolio selection problem from [1]. The portfolio problem is defined and solved analytically in Section 3. Numerical solutions of varying degree of computational effort are calculated in Section 4 and compared to the analytical solution. In Section 5, a few specific problems of financial engineering are solved. Concluding remarks close the paper.

2. A Simple Markovian Approximation

2.1. Optimal Stochastic Control. Consider the stochastic system to be controlled

\[
dX(t) = f(X(t), u(t), t)dt + b(X(t), u(t), t)dW(t)
\]

(1)

where

\[
X = \{X(t) \in X \subset \mathbb{R}^n, t \geq 0, X(0) = x_0 \text{ -- given}\}
\]

is the state process, \(u(t) \in U \subset \mathbb{R}^m\) is the control, \(W(t)\) is a Wiener process, \(f(X(t), u(t), t)\) is a drift, and \(b(X(t), u(t), t)dW(t)\) is diffusion. For the formal treatment of the optimally

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\(^1\)Analytical optimal portfolio rules are known for the HARA (Hyperbolic Absolute Risk-Aversion) family of utility functions that includes isoelastic, exponential and quadratic utility functions, see [9]. However, the explicit solutions to some "practical" problems that would allow for time dependent model parameters are usually beyond the simple quadratures.

\(^2\)This is another class of analytically intractable yet sensible portfolio problems. In principle, constrained policies could be obtained through the Kuhn-Tucker conditions. In practice, their closed forms are unobtainable.
controlled diffusion process refer to [1]. The optimal control rule \( \mu \) that determines the control \( u \) is Markovian

\[
(2) \quad u(t) = \mu(t, X(t))
\]

and chosen so as to maximise a functional \( J \)

\[
(3) \quad \max_u J(0, x_0; u)
\]

where

\[
(4) \quad J(\tau, x; u) = \mathbb{E} \left( \int_{\tau}^{T} g(X(t), u(t), t) dt + s(X(T)) \mid X(\tau) = x \right)
\]

is the profit-to-go function. For Markovian feedback controls the maximum value of (4)

\[
H(\tau, x) = \max_{\mu(\cdot)} J(\tau, x; \mu(\cdot))
\]

satisfies, under appropriate assumptions [1], the HJB equation

\[
(5) \quad \max \{ g(x, u, t) + \mathcal{L} u H(t, x) \} = 0
\]

with the boundary condition

\[
(6) \quad H(T, X(T)) = s(X(T))
\]

where \( \mathcal{L} u \) is the operator

\[
(7) \quad \mathcal{L} u = \frac{\partial}{\partial \tau} + \sum_{i=1}^{n} f_i(x, u, \tau) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} B_{ij}(x, u, \tau) \frac{\partial^2}{\partial x_i \partial x_j}
\]

and where \( f_i \) is the i-th component function of \( f \) and \( B_{ij} \) is the \( i,j \)-th entry of the covariance matrix \( B = bb^T \).

2.2. A Corresponding Markov Decision Chain. A Markov decision chain corresponding to the optimisation problem \( \max(4) \) subject to (1) and (2) is obtained through the following three steps. First, the state equation (1) is discretised in time using the Euler-Maruyama approximation (see [4]). Then, the state space is restricted to a finite dimensional discrete state grid and, finally, the transition probabilities and rewards for these discrete states are specified.

**Euler-Maruyama Approximation.** The approximation scheme is introduced here for a one dimensional process. The extension of the scheme to \( \mathbb{R}^n \) is obvious.

An Euler-Maruyama approximation of process \( X \subset \mathbb{R}^1 \) that satisfies equation (1) is a stochastic process

\[
Y = \{ Y_\ell \in X, \ \ell = 0, 1, \ldots, N \}
\]
satisfying the equation (called the iterative scheme)

\[ Y_{t+1} = Y_t + f(Y_t, u_t, \tau_t) (\tau_{t+1} - \tau_t) + b(Y_t, u_t, \tau_t) (W(\tau_{t+1}) - W(\tau_t)) \]

\( \tau \in \{ 0 = \tau_0 < \tau_1 < \cdots < \tau_N = T \} \) which is a partition of the time interval \([0, T]\).

The indices run \( \ell = 0, 1, 2, \ldots, N - 1 \), the initial and subsequent values are, respectively

\[ Y_0 = X(0) = x_0, \quad Y_\ell = Y(\tau_\ell). \]

For a \( N \)-step time discretisation using a constant time step

\[ \tau_\ell = \ell \delta \quad \text{where} \quad \delta = \tau_{\ell+1} - \tau_\ell = \frac{T}{N}. \]

**Notation.** The discretisation scheme, while intuitively simple, overlays several layers of discretisation: of time, of state, and of noise. We adopt the following conventions.

1. Continuous-time variables: \( x(t) \) (standard); variables in discrete time: \( x_\ell \).
2. Points of the discrete state space (“grid”) \( \bar{x} \in \bar{X}_\ell \).
3. Stochastic processes: \( x \) (bold).

**Discrete State Space.** Equidistant grids will be used for simplicity. The discrete state space for stage \( \ell \) is denoted by \( \bar{X}_\ell \subset \mathbb{R}^1 \). Let the upper and lower bounds of the state grid be

\[ \bar{U}_\ell = \max \bar{X}_\ell \quad \text{and} \quad \underline{U}_\ell = \min \bar{X}_\ell. \]

respectively. A point \( x \in X \) is defined to be within the grid \( \bar{X}_\ell \) if \( \underline{U}_\ell \leq x \leq \bar{U}_\ell \). The collection of the discrete state spaces for all the stages, \( \{\bar{X}_\ell\}_{\ell=0}^N \), is denoted \( \bar{X} \) and called the discrete state space.

**Adjacency.** Heuristically, the scheme approximates a point of \( X \) at stage \( \ell \) by the points of \( \bar{X}_\ell \) which are “adjacent” to it.

1. Two states of \( \bar{X}_\ell \) are **adjacent** if no other state of \( \bar{X}_\ell \) lies between them. In \( \mathbb{R}^n \) two states are **adjacent** if their projections onto each of the \( n \) coordinate axes are adjacent in the sense just defined.
2. Given a point of the continuous state space, \( x \in X \), a pair of states, \( \bar{x}^0 \in \bar{X}_\ell \) and \( \bar{x}^0 \in \bar{X}_\ell \), is **adjacent to** \( x \) if the states are adjacent and \( \bar{x}^0 < x < \bar{x}^0 \).
3. Given \( x \in X \) with \( x \geq \bar{U}_\ell \) define \( \bar{U}_\ell \) to be **adjacent to** \( x \).
4. Given \( x \in X \) with \( x \leq \underline{U}_\ell \) define \( \underline{U}_\ell \) to be **adjacent to** \( x \).
5. Given \( x \in X \) with \( x \in \bar{X}_\ell \) define \( x \) to be **adjacent to** itself.
Transition Probabilities. Consider the stochastic process \( Y = \{ Y_\ell, \ell = 0, 1, 2, \ldots, N \} \) where \( Y_\ell \) is defined through equation (8). For a given control sequence \( u_\ell \) and equidistant discretisation times, the iterative scheme (8) can be abbreviated to

\[
Y_{\ell+1} = Y_\ell + \delta f_\ell + b_\ell \Delta W_\ell
\]

where \( f_\ell \) and \( b_\ell \) denote, respectively,

\[
f_\ell = f(Y_\ell, u_\ell, \tau_\ell) \quad \text{and} \quad b_\ell = b(Y_\ell, u_\ell, \tau_\ell).
\]

The increments

\[
\Delta W_\ell = W(\tau_{\ell+1}) - W(\tau_\ell), \quad \text{for } \ell = 0, 1, 2, \ldots, N - 1
\]

refer to the Wiener process \( W = \{ W(t), t \geq 0 \} \) and are known (cf [4]) to be independent Gaussian random variables with mean and variance:

\[
\mathbb{E}(\Delta W_\ell) = 0 \quad \text{and} \quad \mathbb{E}((\Delta W_\ell)^2) = \delta.
\]

The iterative scheme (11) thus defined is the simplest strong Taylor approximation of an Ito diffusion process (1), see [4]. Now, suppose that at some time \( \tau_\ell \), \( Y_\ell = \bar{Y}_\ell \in \bar{X}_\ell \).

Deterministic process. Assume, for the time being, that there is no noise in the process (11) so, for a given control value \( u_\ell \), the process moves to \( Y_{\ell+1} \) which is defined by:

\[
Y_{\ell+1} = \bar{Y}_\ell + \delta f_\ell.
\]

If there is a pair of states of \( \bar{X}_{\ell+1} \) adjacent to \( Y_{\ell+1} \) then the transition probabilities are assigned using an inverse distance method. Let \( \bar{Y}_{\ell+1}^a < \bar{Y}_{\ell+1}^b \) be the pair of states adjacent to \( Y_{\ell+1} \). Define

\[
h_\ell = \bar{Y}_{\ell+1}^b - \bar{Y}_{\ell+1}^a
\]

and assign the following non-zero transition probabilities

\[
p(\bar{Y}_{\ell}, \bar{Y}_{\ell+1}^a | u_\ell) = \frac{\bar{Y}_{\ell+1}^a - \bar{Y}_{\ell+1}^b}{h_\ell}
\]

\[
p(\bar{Y}_{\ell}, \bar{Y}_{\ell+1}^b | u_\ell) = \frac{\bar{Y}_{\ell+1}^b - \bar{Y}_{\ell+1}^a}{h_\ell}.
\]

A Weak Taylor Approximation. If the Gaussian noise is present in (11) a value of \( Y_{\ell+1} \) is not deterministic. For this situation, the strong Euler scheme (11) will be replaced by a weak Euler scheme (see [4])

\[
Y_{\ell+1} = Y_\ell + \delta f_\ell + b_\ell \Delta \tilde{W}_\ell.
\]
The difference is in \( \Delta \tilde{W}_\ell \), which is a “convenient” approximation of the random increments \( \Delta W_\ell \) of the Wiener process that has similar moment properties to those of \( \Delta W_\ell \). In the portfolio model, we will use an easily generated two-point random variable taking values \( \pm \sqrt{\delta} \) i.e.,

\[
P \left( \Delta \tilde{W}_\ell = \pm \sqrt{\delta} \right) = \frac{1}{2}.
\]

This approximation of the continuously distributed perturbation \( \Delta W_\ell \) by a two-value noise is of course arbitrary. However, it is sufficient for the approximating solutions’ convergence. One can obviously use other more realistic discrete representations of \( \Delta \tilde{W}_\ell \) e.g., it can be modelled as a three-point distributed random variable \( T_\ell \) with

\[
P \left( T_\ell = \pm \sqrt{3\delta} \right) = \frac{1}{6}, \quad P \left( T_\ell = 0 \right) = \frac{2}{3}.
\]

No matter how simple or complex these approximations are, they should preserve the original distribution’s first and second moments and depend on the partition interval’s length. The latter feature guarantees that, for all such approximations, the smaller \( \delta \) the less diffuse the states become, to which the process transits.

For the noise representation (16), the definition of the transition probabilities in the stochastic case is only slightly different from (13), (14). Let \( Y_{\ell+1} \) be determined through (12). The noise discretisation method means that for \( \delta > 0 \) the process reaches, at \( \ell + 1 \):

\[
Y_{\ell+1}^- = Y_{\ell+1} - b_\ell \sqrt{\delta} \quad \text{with prob.} \frac{1}{2}
\]

\[
Y_{\ell+1}^+ = Y_{\ell+1} + b_\ell \sqrt{\delta} \quad \text{with prob.} \frac{1}{2}.
\]

If there are two adjacent states to \( Y_{\ell+1}^- \) and \( Y_{\ell+1}^+ \) then apply the inverse distance method as in (13), (14) but weight the two probabilities by \( \frac{1}{2} \). Thus, for example, if \( Y_{\ell+1}^- \not\in X_{\ell+1} \) but there exist \( Y_{\ell+1}^{-\ominus} < Y_{\ell+1}^- \) in \( X_{\ell+1} \) adjacent to \( Y_{\ell+1}^- \) then the transition probabilities are

\[
p(Y_\ell, Y_{\ell+1}^{-\ominus} | u_\ell) = \frac{1}{2} \frac{Y_{\ell+1}^- - Y_{\ell+1}^{-\ominus}}{h_\ell}
\]

\[
p(Y_\ell, Y_{\ell+1}^{-\ominus} | u_\ell) = \frac{1}{2} \frac{Y_{\ell+1}^- - Y_{\ell+1}^{-\ominus}}{h_\ell}
\]

where \( h_\ell = Y_{\ell+1}^- - Y_{\ell+1}^{-\ominus} \). If any of the states \( Y^{-\ominus}, Y^{+\ominus}, \ etc. \) overlaps another, the respective probabilities have to be summed up.

As is evident, the above discretisation method is very simple and intuitive. However, as noted, it preserves the first two moments of the original distribution so that the overall discretisation
scheme is weakly consistent.\footnote{Kushner lists in [8], page 1002, three conditions for consistency of an approximation scheme. Conditions “1” and “2” (about continuity of the Markov chain expected value and variance) are easily satisfied:

1. $\mathbb{E}(y_{t+1}^h - y_t^h) = \delta f_t$;
2. $\mathbb{E}[(y_{t+1}^h - y_t^h)^2] = \delta b_t^2$;
3. $|y_{t+1}^h - y_t^h| = O(\sqrt{\delta})$.

Moreover, consistency fails along the boundary of the discrete state space so the scheme is locally weakly consistent. This is not surprising since it would be impossible for a system constrained to lie within a finite space to follow the behaviour of a system which is not similarly constrained at the points where the constraints become active. However, this feature is common to all approximation schemes of this kind.}

**Constraints.** It has to be borne in mind that the discretisation of a constraint is always sensitive to the choice of the discretisation steps $\delta$ and $h$, and has to be dealt with “carefully”. E.g., a local constraint $v(t) > a$ for $t \in [t_1, t_2]$ can be meaningfully translated as $v_t > a$ only if $\delta$ is small in comparison to $[t_1, t_2]$ and the state grid step $h < |a|$. See inequality (35) for an example of how a state variable constraint can be represented in discrete time.

**Transition Rewards.** Let the control strategy be Markovian (2) and action at state $\ell$ computed as

$$\mathbf{u}_\ell = \mu(\ell, \mathbf{Y}_\ell), \quad \mathbf{Y}_\ell \in \mathbf{X}_\ell, \; \ell = 0, 1, \ldots N - 1.$$  

Recalling (4), note that for the approximating problem, the decision maker receives a reward that depends on the state at stage $\ell$ and on the action $\mathbf{u}_\ell$

$$\gamma(\mathbf{Y}_\ell, \mathbf{u}_\ell, \ell) = \delta g(\mathbf{Y}_\ell, \mathbf{u}_\ell, \ell),$$  

$\ell = 0, 1, 2, \ldots N - 1$. The overall reward for the Markov decision chain $\mathbf{Y}$, starting from $\mathbf{Y}_0 = x_0 \in \mathbf{X}_0$ and controlled by $\mathbf{u} = \{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}\}$ can be determined as

$$J(0, \mathbf{x}_0; \mathbf{u}) = \mathbb{E} \left( \sum_{\ell=0}^{N-1} \gamma(\mathbf{Y}_\ell, \mathbf{u}_\ell, \ell) + s(\mathbf{Y}_N) \mid \mathbf{Y}_0 = x_0 \right)$$

Finally, the problem:

$$\begin{align*}
\max_{\mathbf{u}} & \quad J(0, \mathbf{x}_0; \mathbf{u}) \\
\text{subject to} & \quad \mathbf{Y}_{t+1} = \mathbf{Y}_t + \delta f_t + b_t \mathbf{w}_t,
\end{align*}$$

with the transition probabilities defined as above is the Markov decision chain approximating the original continuous-time optimisation problem of Section 2.1. For convenience, we use the notation $J(\delta, \mathbf{X}) \equiv J(0, x_0; \mathbf{u})$ where $\mathbf{u}$ is a maximiser in the above optimisation.
2.3. **Computational Complexity.** There are two crucial parameters for the solution method outlined above: the number of states and the number of time steps. One expects that increasing these numbers would improve the solution’s accuracy. However, the computation time also increases. Recent papers [14] and [15] report mitigating the curse of dimensionality for a certain subclass of Markov decision chains through use of randomisation. The Markovian approximation defined in this paper leads to a similar conclusion. Following [6] a result of this nature will be proved.

**Claim.** The computation time needed for the solution of the Markov decision chain (25) increases approximately linearly in both the number of states and the number of time steps.

**Proof.** Suppose a solution is computed by backward induction for a state in stage \( \ell \) and that the solution from stage \( \ell + 1 \) onwards has already been determined. The time required to compute the optimal decision for the current state is largely independent of both the number of time steps and the number of states. Its independence of the number of states is a consequence of the approximation scheme scanning only the adjacent states in the next stage. Doubling the number of states means that twice the time is taken for each stage and the computation time doubles. Doubling the number of time steps leaves the computation time for each stage fixed but doubles the number of stages and hence the computation time doubles. This exact linear relationship reported in [6], see Figure 1, is spoiled by the vagaries of the computation time of the numerical maximisation, as well as the load dependence of the computer performance (see Figure 1 right panel).

![Figure 1. Computational complexity.](image-url)
An array of test problems was solved in [6] (and [7]) using a similar Markovian approximation method. The difference was in the noise approximation method. So, only deterministic problems’ solutions there reported are directly relevant for this paper. The approximating solutions closely followed the optimal ones. However, the test problems were “easy” in that they contained a quadratic cost component (they were not linear-quadratic though). Here, the method will be used to solve a classical portfolio selection problem (see [1]), which is generically stochastic and non linear-quadratic.

3. A Portfolio Selection Model


The stock portfolio consists of two assets, one “risky” and the other “risk free”. If the price per share of the risky asset \( p(t) \) changes according to

\[
 dp = (\alpha dt + \sigma dw)
\]

while the price \( q \) per share for the risk free asset changes according to

\[
 dq = qr dt
\]

then the wealth \( x(t) \) at time \( t \in [0, T] \) changes according to the following stochastic differential equation

\[
 dx = (1 - u_1)rx dt + u_1 x(\alpha dt + \sigma dw) - U_2 dt.
\]

Here, \( w \) is a one-dimensional standard Brownian motion, and \( r, \alpha, \sigma \) are constants with \( r < \alpha \) and \( \sigma > 0 \). The symbol \( u_1(t) \) (respectively, \( 1 - u_1(t) \)) denotes the fraction of the wealth invested in the risky (respectively, risk free) asset at \( t \) and \( U_2(t) \) is the consumption rate. The agent’s objective is to find an optimal two-dimensional strategy \( u = [u_1(x), U_2(x)] \), such that

\[
 0 \leq u_1(t) \leq 1, \text{ and } U_2(t) \geq 0,
\]

and which maximises the expected discounted total utility

\[
 J(0, x(0); u) = \mathbb{E} \left( \int_0^T e^{-\varrho t} U_2(t)^\gamma dt \bigg| x(0) = x_0 \right)
\]

given the discount rate \( \varrho > 0 \) and assuming that \( [U_2(t)]^\gamma \) is the agent’s utility function, with \( 0 < \gamma < 1 \). Here no value is assigned to wealth at \( T \) while \( x_0 \) is the wealth at the initial time \( 0 \). The problem to maximise (29) subject to (27) is clearly one of the class described in Section 2.1.
3.2. **The Optimal Solution.** The Hamilton-Jacobi-Bellman equation can be solved for the optimal value function in the following form

\[ H(\tau, x) = g(\tau)x^\gamma. \]

Function \( g(\tau) \) can be integrated and equals

\[ g(\tau) = e^{-\varphi \tau} \left[ \frac{1 - \gamma}{\theta - \nu \gamma} \left( 1 - e^{-\frac{\theta - \nu \gamma}{\theta - \nu \gamma}(\tau - \gamma)} \right) \right]^{1 - \gamma} \]

where

\[ \nu = \frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma)} + r. \]

The optimal investment and consumption strategies \( \hat{u}_1 \) and \( \hat{U}_2 \) can be computed as

\[ \hat{u}_1 = \frac{\alpha - r}{\sigma^2(1 - \gamma)} \]

\[ \hat{U}_2(\tau, x) = [e^{\varphi \tau}g(\tau)]^{\frac{1}{1 - \gamma}} x. \]

In this example, only \( \hat{U}_2 \) is a (linear) function of wealth while \( \hat{u}_1 \) is constant. Notice also that the above solution is "internal" in that both constraints (28) will be satisfied for some parameter set. In particular \( \hat{u}_1 \leq 1 \) if \( \alpha - r \leq \sigma^2(1 - \gamma) \).

4. A CALIBRATED MODEL

4.1. **The reference solution.** Suppose an agent with an original wealth of \( x_0 = $100,000 \) wants to maximise their satisfaction during the coming \( T = 10 \) years. The instantaneous satisfaction is measured by \( \sqrt{U_2(t)} \). The risk free asset price drift is \( r = .05 \) and that of the risky asset is \( \alpha = .11 \) with the volatility \( \sigma = .4 \). The agent’s discount rate is \( g = .11 \).

For these parameter values, the agent’s expected discounted total utility is \( \bar{J} = g(0)\sqrt{100000} = 723.09 \). Figure 2 presents the optimal strategies; the expected wealth and consumption rate time profiles are shown in Figure 3.

Figure 4 shows ten wealth and strategy time profiles, which correspond to ten noise realisations \( dw(t), t \in [0,T] \), obtained from a random number generator. The average total discounted utility of these 10 portfolios is 751, which is more than the theoretical \( \bar{J} = 729 \). However, it is evident (e.g., from the large variations of consumption realisations) that the optimal portfolios’ performance has a large variance. An estimate of the optimal utility standard deviation was computed for 600 noise realisations and the integration step .025. The mean utility was 719.7, which was 99.5 % of the theoretical expected optimal performance; the corresponding standard deviation was 165 giving a coefficient of variation 0.23.
Figure 2. Optimal (reference) strategies.

Figure 3. Optimal expected wealth and consumption rate time profiles.
4.2. Numerical Solutions. The SOCSol [19] suite of Matlab functions was used to optimise a portfolio from Section 3. Most of the problem transformation (e.g., from a continuous time and space formulation to a discrete model) is taken care of by the software.

Constraints. The constraints have to be dealt with “manually”. The local instantaneous constraints on the controls $u_1(t)$ and $U_2(t)$ (28) can be immediately expressed in discrete time as

\begin{equation}
0 \leq u_{1,t}, \quad U_{2,t} \geq 0.
\end{equation}

The portfolio admissibility condition, which in the continuous version amounts to $x(t) \geq 0$, $\forall t \in [0,T]$, [3], cannot however be directly replaced by $x_{t} \geq 0$. This is (mainly) because $x_t$ is expressible in terms of $u_{1,t-1}$ and $U_{2,t-1}$ yet we need a condition valid for time $t$. It appears (from [9]) that the best discrete-time counterpart of $x(t) \geq 0$ is

\begin{equation}
x_t \left(1 + \delta(r + u_{1,t}(\alpha - r)) \right) - \delta U_{2,t} \geq 0
\end{equation}

where $\delta$ is the time discretisation step (see Section 2.2). It has to be borne in mind that (35) is an approximation to the portfolio admissibility condition and that it depends on the time discretisation step.

Technical hints. A cautionary remark about numerical optimisation is in place. Most optimisation methods work (much) more efficiently if the solution vector components are of comparable magnitudes. This is not the case of the control variables $u_1$ and $U_2$. Indeed, $u_1$ is bounded
between 0 and 1 but, $U_2$ is practically unbounded from above, see Figure 2. This caused some (not insuperable) difficulties in [5] in obtaining accurate approximating solutions. In this paper, such difficulties were avoided through re-scaling of the model. It follows from (33) that $U_2(t)$ is linear in the state $x(t)$. All $U_2(t)$ were then replaced by $u_2(t)x(t)$ in the optimisation problem max (29) subject to (27). Consequently, the numerical routines were looking for $u_2(t)$ that was not greater than 12 for most of the cases solved. Moreover, because of the above transformation the strategy graphs will no longer be linear as in Figure 2 (low panel) but horizontal (as in Figure 7). The re-scaling required knowledge of the solution (33). In general, one has to expect that the optimal solution is a function of the state. However, assuming a linear feedback would be a guess, which could be made in an attempt to deal with controls of comparable values, or justified by the manager's knowledge of the problem to solve.

Important software control parameters are the time discretisation step $\delta$ and the state space grid width $h$. To get an idea of their range values, necessary for an accurate approximation, a deterministic portfolio control problem ($T = 2$) was solved: first analytically, then the discretised model solutions were computed. Figure 5 shows the results.

The plot coordinates are the time discretisation step $\delta$ and a utility measure. The horizon is $T = 2$; the remaining model parameters are as in Section 4.1.

The point denoted “*” on the $y$-axis at 422 is the continuous model optimal utility. The discrete time model utility values converge toward this point as $\delta \to 0$. Notice that they are greater than

![Figure 5. Discretised model utility realisations.](image-url)
the continuous model utility. This is because (in the rectangular method) the integration error grows in $\delta$. The points denoted “+” correspond to utility realisations of a model discretised both in time and space (Markov chain). It is clear from the figure that reasonable utility approximations can only be obtained for $\delta \leq .1$ and $h \leq 500$.

The impact of the length of the time step $\delta$ on the solution accuracy in a stochastic model is shown in Figure 6.

**Figure 6.** Strategy convergence.

Consider time $\ell$ ($\ell = 0, 1...N-1$) and $u_{1,\ell}$ to be applied at this time. Assuming that the choice of $U_{2,\ell}$ is made optimal

$$u_{1,\ell} = \arg \max \left( \delta \sqrt{U_{2,\ell}} + e^{-\sigma^2 g(\ell + \delta)} \mathbb{E} \sqrt{x_{\ell+\delta}} \right)$$

(36)

$$= \arg \max \left( \mathbb{E} \sqrt{x_{\ell+\delta}} \right),$$

see (30). The expected value in (36) was computed using a Taylor series (second order) expansion and presented as a function of strategy $u_1$ in Figure 6. The strategy domain was “extended” beyond the feasible range $[0, 1]$ to show the utility measure shapes. Notice that, for the feasible $u_1 \in [0, 1]$, the utility measures would all look flat.

The vertical line $u_1 = .75$ shows where the utility maximum “should” be, for it is known from Figure 2 that the optimal strategy is $u_1 = .75$, independent of time. It is clear from the figure that the discretised model strategy $u_{1,\ell}(\delta)$ converges to the optimal strategy as $\delta \to 0$. Again, any reasonable approximation requires $\delta \leq .1$. 
Another point to remember is that the time step \( \delta \) should be "large" relative to the process dynamics i.e., \( |f \delta + \sqrt{\delta} | > h \). Otherwise, the process would not be able to move beyond the closest adjacent state. On the other hand, \( \delta \) should be small for high accuracy of the approximating solutions (as shown above; also, see Footnote 3, page 8). These conflicting requirements can be resolved for small \( h \). This implies a dense state grid, which will increase the computation time.

The convergence. There are various ways in which the goodness of a numerical solution can be evaluated. The most “objective” one would perhaps be to look at the average discounted total utility \( J \) generated by the application of an approximated optimal numerical solution to the continuous model. However, as evident from Figure 4, the portfolio performance is very “volatile” and the standard deviation of utility distribution is large. Therefore, using \( J \) thus computed is difficult to judge which solution is best.

We will first evaluate the convergence by comparing the approximating policy profiles (Figures 7 - 12), to the optimal ones (Figure 2). Remember that because of the model re-scaling, if \( U_2 \) is linear (Figure 2) \( u_2 \) has to be horizontal. Then, we will generate a few realisation profiles to compare them to those of Figure 4 and, eventually, we will compute the corresponding utility distribution.

![Approximating strategies](image)

**Figure 7.** Approximating strategies for \( t = 0 \) \( (\delta = .2) \).

Examine the policy rules shown in Figures 7 - 12. The bold dotted lines correspond to optimal strategies (compare Figure 2). One can see that the strategy convergence is more difficult to achieve for later times \( (t = 9) \) than at the beginning of the horizon \( (t = 0) \).
Indeed, the gap between the optimal strategy and the approximating strategies for $t = 0$ is narrow for $\delta = .2$ and closes for $\delta = .1$ (for reasonably small $h$) whereas, for $t = 9$, it narrows down only for smaller $\delta$s, see Figures 11 and 12. This is to be expected because the optimal $\hat{U}_2(T) = \infty$, and $\hat{d}_2(T) = \infty$ (see (33) and (31)), which is impossible to reproduce numerically.

Figure 13 shows the wealth and strategy realisations for $\delta = .05$ and $h = 100$. They look very similar to the optimal ones in Figure 4. The simulation of 2000 noise realisations and the
FIGURE 10. Approximating strategies for $t = 9$ \hspace{2em} ($\delta = .1$).

FIGURE 11. Approximating strategies for $t = 9$ \hspace{2em} ($\delta = .05$).

application of the approximating policy rules computed for the same parameters (i.e., $\delta = .05$ and $h = 100$) resulted in the utility distribution (integrated with the time simulation step equal to .025) shown in Figure 14.

The mean discounted utility is $\hat{J} = 715.4$ (98.9% optimal) and the corresponding standard deviation is 161. However, the portfolio performance as judged by index $\hat{J}$ for other approximating rules (e.g., $\delta = .05, .02$ and $h = 500, 100$) was comparable ($\hat{J} \in [701, 720]$ with standard deviations
\begin{figure}
\begin{center}
\includegraphics[width=\textwidth]{figure12}
\end{center}
\caption{Approximating strategies for $t = 9$ \hspace{1em} ($\delta = .02$ \textit{``small''}).}
\end{figure}

\begin{figure}
\begin{center}
\includegraphics[width=\textwidth]{figure13}
\end{center}
\caption{Approximated wealth, investment and consumption rate realisations.}
\end{figure}

$\in [159, 162]$. However, using $\delta = .5$, $h = 10000$ that is clearly a \textit{``bad''} policy, resulted in $J = 124.13$ with the standard deviation equal 100.
Averaging the utility over more realisations and diminishing the simulation step could help to improve the utility variance estimate. However, the improvement would not be substantial, as the portfolio performance is highly “volatile” whether optimal (Figure 4) or not (Figure 13).

5. Portfolio Model Modifications

5.1. Time Varying Parameters. Suppose now that the agent expects the volatility coefficient \( \sigma \) to vary as follows

\[
\sigma(t) = \underline{\sigma} \left(1 - 0.09 \cos \left(\frac{2t}{\pi}\right)\right)
\]

where \( \underline{\sigma} = .4 \). This means that the volatility \( \sigma \) used in Sections 3 and 4 was an average value. Now, the volatility will rise from 36.4 \% to 43.6 \% in the middle of \( T \), and then drop. The agent would like to know whether this information should change their investment strategy or not.

Figure 15 reveals the modified strategy obtained as a solution to the discretised portfolio problem with \( \delta = .05 \) and \( h = 500 \). As expected\(^4\), the agent will invest more (i.e., above .75) when the volatility is low. The solution is “exact” in that we can read how much one has to

\(^4\)See footnote 6.
invest at each time; it also follows the closed form solution, which is

$$
\bar{u}_1(t) = \frac{\alpha - \gamma}{\sigma^2(t)(1 - \gamma)}.
$$

The optimal consumption strategy appears unchanged. This is a result of the parameter choice, for which the varying parameters equivalent of function $g(t)$ does not differ substantially (numerically) from (31).

![Figure 15. Cosine modified strategies.](image1)

![Figure 16. Cosine modified realisation paths.](image2)
The wealth and strategy sample paths for the cosine volatility problem are shown in Figure 16. The discounted total utility is 664 (std=158; giving a coefficient of variation .24).

In summary, the agent should modify their investment strategy once the information about a volatility scenario becomes available. In a similar way, a portfolio problem with a time dependent interest rate (or other parameters) can be solved.

5.2. Constrained Policies. A portfolio manager might have an a priori belief that their investment should not exceed a certain wealth percentage. Such a constraint can easily be allowed for in the above numerical optimisation procedure.

Suppose that the permissible investment level is $\bar{\alpha}_1 = .5$. Figure 17 reveals the modified strategy. Not surprisingly, the agent is expected to invest at the constraint. However, this does not affect the consumption strategy. The new discounted total utility is 717 and the standard deviation equals 108, which is substantially less than under the unconstrained regime. The wealth and strategy sample paths are shown in Figure 18. The conservative investment strategy results in much less “volatility” in consumption and wealth as observed by eye and measured by the utility standard deviation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Constrained strategies.}
\end{figure}

An impact of the investment constraint on the expected performance can be observed in more detail in Figure 19. Two histograms of the discounted total utility realisations are presented. The dark shadowed one corresponds to the unconstrained policies (compare Figure 14). The light grey histogram represents the constrained policy performance. Evidently, the constrained policy guarantees more “secure” performance (standard deviation = 108 vis-à-vis 166 of the
unconstrained policy). However, the unconstrained policy brings a (marginally) higher utility value. More computations of that kind would generate the “efficient boundary”.

In a similar way, another portfolio problem, in which some minimal (or maximal) consumption rate is given could be solved.
5.3. **Pension funds.**

5.3.1. *An optimal control problem.* A practical problem of financial engineering is one in which an agent pays an amount $x_0$ to a pension fund, to be repaid by a lump sum $\tilde{x}_T$ at time $T$. The latter is a result of an investment policy $u_1(x)$ adopted by the fund’s manager.

The manager’s policy depends on his or her objective function, which could be the maximisation of an expected value, the minimisation of risk to obtain a target amount, *etc.* Once the objective function is revealed, the manager’s policy can be computed as a solution to a stochastic optimal control problem associated with the objective function. The problem solution will routinely comprise an optimal decision rule $u_1(x), U_2(x)$ and a Monte-Carlo simulated distribution of $x_T$. Knowing the former is crucial for the manager to control the portfolio. The latter is “practical” in that it tells the pension buyer what they can, or should, expect as $x_T$.

Knowing the distribution of $x_T$ also helps the manager. It gives them an idea of what probabilities, or risks, are associated with obtaining a particular realisation of the objective function. For example, the distribution may suggest that, for every $x_0$ there is a probable terminal value $\overline{x}_T$, which the manager may choose to advertise as the pension target (subject to legislative constraints on financial advertising).

5.3.2. *Expected value maximisation.* We will first solve a pension fund problem for the expected value criterion, as follows. In (4), set $g(X(t), u(t), t) = 0$, $s(T(t)) = x(T)$ and suppose that the management fee is 2% of $x(t)$. This means that we deal with a non HARA objective function and want to optimise it in $u_1(x)$ with $U_2(t) = .02x(t)$.

Using the Markovian approximation approach as in Section 4.2, with the same model parameters *i.e.*, $T = 10, r = .05,$ *etc.*, generates a rather trivial optimal strategy: $u_{1,t} = 1, U_{2,t} = .02x_t$ for positive states and times. Applying the strategy to different initial outlays $x_0$ generates the following final fund yield location and spread measures, see Figure 20. This figure as well as the following histograms were obtained for 1200 realisations. The figure’s horizontal axis represents the initial outlay $x_0$ while the values of the location and spread measures of $x_{10}$ (mean, median and standard deviation, respectively) are presented on the vertical axis.

The figure tells us, among other things, that an initial deposit of, for example, $40,000$ corresponds to the expected terminal lump sum of about $100,000$. However, the median yield is significantly below the mean. This indicates that the fund distribution is skewed, which is even more evident from the histogram shown in Figure 21 (upper panel).
FIGURE 20. Yield location and spread measures.


This histogram also shows us that the probability of earning less than the “secure” revenue i.e., one earned by investing in the secure asset only:

\[ 40,000 \exp \left\{ (r - \text{“management fee”})10 \right\} = 53,994 \]
is more than .5. To see this and prove the subsequent claims integrate the area under the
histogram from zero to 53,994 and 40,000, respectively.

It is even fairly probable (with probability >.4) that the final payoff will be less than the initial
outlay $x_0 = 40,000$. Evidently, using a policy that maximises the expected yield is a very risky
strategy of managing a portfolio. A couple of alternative policies will be analysed in the next
subsection.

To quantify a risk level associated with this policy, two popular risk measures, Value-at-Risk
(VaR) and Conditional Value-at-Risk\(^5\) (CVaR) will be approximately calculated from the his-
togram. The measure explanation and calculations are moved to the Appendix. Here, however,
we notice that,
\begin{itemize}
\item for a specified probability level $\beta$, the $\beta$-VaR of a portfolio is the lowest amount of loss $\alpha$ whose
probability does not exceed $\beta$. For the policy maximising the expected yield earned by an initial
outlay $x_0 = 40,000$ and the loss defined as a yield below the outlay

\begin{equation}
.9-\text{VaR} \approx 31,000.
\end{equation}

see Appendix.
\item for the same probability level $\beta$, the $\beta$-CVaR is the mean loss above the amount $\alpha$. For the
same policy,

\begin{equation}
.9-\text{CVaR} \approx 35,000.
\end{equation}

These measures disqualify the policy of maximising the expected yield as a fund manager’s ob-
jective function. No manager would accept such a high risk in controlling a pension fund.

5.3.3. Alternative strategies. Suppose now that the manager will use a “constrained” strategy:
invest 50\% in the risky asset $u_{1,t} = .5$ (and pay the respective management fee $U_{2,t} = .02x_t$).
Such a policy is clearly “non optimal” for the expected value objective function. However, we can
see from Figure 21 (lower panel) that, for the same initial outlay as before (i.e., $x_0 = 40,000$),
the yield distribution is more concentrated and less skewed than the previous (unconstrained)
one shown in the upper panel. The mean for this portfolio is $84,100$ (median=58,710) and the
standard deviation diminishes to $45,563$ (from $168,000$ for the unconstrained policy). Overall,
the risk of performing worse than investing in the secure asset alone is much less here than under
the unconstrained (“optimal”) strategy. The risk measures are

\begin{equation}
.9-\text{VaR} \approx 13,000
\end{equation}

\begin{equation}
.9-\text{CVaR} \approx 17,000.
\end{equation}

\(^5\)See [13] for a static portfolio analysis based on VaR and CVaR.
Although these values are less than for the case of the expected yield maximisation (see (39), (40)), they are still high and probably not acceptable to many fund managers.

We will design a couple of alternative policies (called “cautious”) each determined as a solution to a stochastic optimal control problem with the utility (objective) function $s(x_T)$ given as follows

$$J(0, x(0); u^*) = \max_u \mathbb{E} \left( s(x_T) \middle| x(0) = x_0 \right)$$

where

$$s(x_T) = \begin{cases} (x_T - \bar{x}_T) \gamma & \text{if } x_T \geq \bar{x}_T, \\
-(\bar{x}_T - x_T)^p & \text{otherwise} \end{cases} \quad 0 < \gamma < 1, \ p > 1.$$  

This criterion reflects the client’s (and manager’s) wish to dispose of sufficient funds to meet the target $\bar{x}_T$. The reward for exceeding $\bar{x}_T$ is moderate ($0 < \gamma < 1$) while the punishment for not reaching it might be made substantial ($p > 1$). Two policies will be computed for different combinations of $\gamma$ and $p$.

Suppose that the wealth target is $\bar{x}_{10} = 100,000$. The rest of the problem parameters are as before i.e., $T = 10$, $u_2 = .02$, $r = .05$, etc. Also notice that the target wealth will be reached (with certainty) if $x_0 = 74,081.82$ is invested in the secure asset. We will consider initial deposits less than this amount.

Two optimal investment policies resulting from the solution to (43), (44) for different values of $\gamma$ and $p$ are shown in Figure 22. The horizontal axis in each panel is the wealth at $t$ ($x_t$) and the vertical axis is $u_t$. Each curve in Figure 22 corresponds to a strategy for a different time. The times represented in the upper panel are: the beginning of the investment period ($t = 0$), the middle time ($t = 5$) and a final time ($t = 9.9$). Both strategy lines in the lower panel are for $t = 5$.

In the upper panel, the three curves represent a cautious policy obtained for a client whose preferences are reflected by $\gamma = \frac{1}{7}$ and $p = 2$. Here, the preferences tell us about a client’s “quadratic” fear of not reaching $\bar{x}_{10}$ and a “square-root” enjoyment from exceeding it.

The lower panel shows two strategy lines for $t = 5$. The dashed-dotted line is as in the upper panel. The solid line corresponds to a client’s preferences reflected by $\gamma = \frac{9}{10}$ and $p = \frac{3}{2}$. Here, the client’s attitude is also cautious but more relaxed about not meeting the target $\bar{x}_{10}$ and more enjoying exceeding it than that of the previous client.

The policy lines get higher and steeper as $x_0$ falls and the time-to-go shortens In particular, for the part of the graph where it is impossible to meet the target $\bar{x}_T$ by investing in the riskless asset only (i.e., before each line hits 0.), each line representing a later time policy dominates the
lines that correspond to an earlier policy. This is an interesting solution: if there is a shortage of funds or time, the manager commits more funds to the risky investment than when he (or she) has more wealth and/or time. In other words, the further the investor is from the state from which meeting $x_{10}$ is certain (e.g., $x_0=74,082$ at $t = 0$), the higher the investment is in the risky asset $u_1$.

The lower panel of Figure 22 shows that the optimal policies of the “cautiously relaxed” customer are bolder than those of their “cautious” counterpart. This means that at a given state $(x_t, t)$ the former will be happy to commit more funds to the risky investment than the latter.

The usefulness of the strategies obtained as solutions to a utility maximisation problem (43), (44) for pension fund management can be assessed from the histograms presented in Figures 23 and 24.

The two strategies were each used to manage three initial payments $x_0$ of:

- $40,000$, which is the amount that generated $Ex_{10} \approx 100,000 = \bar{x}_{10}$ under the expected yield maximisation policy (in Section 5.3.2);
- $73,500$, which appears to be the maximal “reasonable” initial outlay in that any deposit above
$74,082 generates $100,000 if invested in the secure asset;
• an arbitrary deposit of $60,000.

The final yield spreads are represented in Figures 23 (upper panel) and 24 by grey and black histograms corresponding to the cautious and the cautiously relaxed polices, respectively. The histograms become dark grey if they overlap.

Figure 23 shows the results of investing $40,000. The lower panel of this figure shows the yield distribution when the expected yield maximisation policy is used (compare Figure 21’s upper panel). In the upper panel, cautious policies’ outcomes are presented. One can see that the type of skewness, which the cautious policies generate, helps the manager to form an acceptably strong expectation of a satisfactory final payoff. For example, for the current set of data, we can say that $x_{10} \geq 70,000$ has the probability about .75, for the cautiously relaxed policy. Under the expected yield maximisation policy, the probability of achieving this result is less than .4.

It is also easy to see that the risk measures VaR and CVaR are very small for all three outlays. For a loss defined as the difference between the initial outlay and the final payoff it is virtually zero for the second outlay, see Figure 24. This speaks well about the applicability of these policies. For the current parameter set, the cautiously relaxed strategy appears attractive in that it generates lower risk values than any other strategy (considered in this paper) for the same initial deposit.
A further analysis of the final yield distributions, calculated for several initial deposits $x_0 \in [60,000, 74,082]$, could suggest a particular $x_0$ that the manager would accept in return for a ten year “bond” $\bar{x}_{10}$.

A different optimal control problem with $g(X(t), u(t), t) \neq 0$ and $s(X(T)) = e^{-\theta T} x(T)$ i.e., one in which a combination of the final wealth and the utility from consumption is maximised, could also be solved using the above approximation method.

6. Conclusion

A discretisation method useful for Markovian approximations of finite-horizon continuous-time stochastic optimal-control problems has been described. An optimising algorithm has been developed (see [19]) and applied to solve a portfolio selection problem. For the calibrated models, an overall agreement between the analytical (where obtainable) and approximating solutions was noticed. However, a large variance of utility realisations was observed. It was also noticed that the variance diminished for constrained optimal solutions.

In the example with variable volatility, no difference was reported in consumption patterns between periods of low and high volatilities. There were, however, differences in the investment schedules: as expected⁶, the volatility troughs triggered higher investment levels. A pension fund management example illustrated the use of the method for a non HARA objective function; it also

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⁶There are situations where the conclusion could have been the opposite, see [18].
highlighted the insufficiency of a mean value as an optimisation criterion. Alternative strategies were computed with very low VaR and CVaR.

All solutions were practical in that they could be applied to real life situations describable by the portfolio model. The method is ready to compute solutions to other scenarios of the model parameters (including a variable discount rate, etc.).

Some optimisation runs (on a Pentium II PC) lasted up to 30 hours (for $\delta = .02, h = 100$). However, an “intelligent” state space search (along the lines of [12] or [17]) could be implemented to accelerate the algorithm convergence.

Appendix A. Computation of .9-VaR and .9-CVaR

In Section 5.3.2 two risk measures were briefly introduced: Value-at-Risk and Conditional Value-at-Risk. Here, they will be defined and calculated for the expected yield maximisation policy applied to an initial outlay of $x_0 = 40,000$.

Both measures play a role in helping to choose a policy that reduces the risk of high losses, see [13]. If we say that a portfolio management policy generates a $\beta$-Value-at-Risk ($\beta$-VaR) equal to an amount $\tilde{\alpha}_\beta$ we mean that the probability of a loss equal to or less than $\tilde{\alpha}_\beta$ is $\beta$. In other words, $\beta$-VaR = $\tilde{\alpha}_\beta$ iff

$$\tilde{\alpha}_\beta = \min \{ \alpha \in R : P(\text{loss} \leq \alpha) \geq \beta \} .$$

There have been studies applying VaR to optimise static portfolio problems. However, as a mathematical operator, VaR lacks subadditivity and convexity. This hampers, among other things, using minimisation methods based on approximations. An alternative measure is suggested in [13]. This is Conditional Value-at-Risk which has better mathematical properties than VaR, see *ibidem*. If a portfolio management policy generates a $\beta$ Conditional Value-at-Risk ($\beta$-CVaR) this means that the expected loss above $\tilde{\alpha}_\beta$ equals $\beta$-CVaR.

These two risk measures will be now computed for the expected yield maximisation policy applied to manage an initial outlay of $40,000$ (see (39) and (40)). Assume the probability level $\beta = .9$.

Figure 25’s upper panel is the same as in Figure 21 and shows the distribution of final yields $x_{10}$ corresponding to $x_0 = 40,000$ managed through the above policy. The lower panel of the figure represents the loss distribution associated with this policy. The loss distribution was obtained from the upper panel by “inverting” it and shifting to the right. The shift was by 40,000 which reflects a somehow arbitrary decision that any final yield below the initial outlay (here, $x_0 = 40,000$) is a loss. The lightly shadowed area equals .1 and cuts off the .9-VaR=$\tilde{\alpha}_9 \approx 31,000$. The mean value of the loss under the area to the right from $\tilde{\alpha}_9$ is .9-CVaR$\approx 35,000$. Notice that
it follows from the definition of these two measures that

$$\beta-\text{VaR} \leq \beta-\text{CVaR}$$

which ensures that portfolios with low CVaR must also have low VaR.

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